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ORIGINAL PAPER

Optimal error bound and generalized Tikhonov regularization for identifying an unknown source in the heat equation

Ailin Qian · Yan Li

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Abstract In this note we prove a stability estimate for an inverse heat source problem. Based on the obtained stability estimate, we present a generalized Tikhonov regularization and obtain the error estimate. Numerical experiment shows that the generalized Tikhonov regularization works well.

Keywords Inverse problems · Ill-posed problems · Generalized Tikhonov regularization · Stability estimate · Error estimate

1 Introduction

In many engineering contexts, there are many inverse heat source problems for heat equation. The problem of this kind arises in many important applications in practice, e.g. With development of society and economics, groundwater pollution has become a serious threat to the environment. The government has to take some measures to prevent the groundwater from further contaminations. But the cost of cleanup for polluted aquifers is staggering, and in many cases it is hard to identify which companies are responsible for the contamination, due to lack of tools to discover the pollution sources. So, it is necessary to try to give more concrete information of the characteristics (location, magnitude, and duration of activity) of specific groundwater pollution sources (see [4]). As we know, most attempts at quantifying contaminant transport rely

A. Qian (✉)
School of Mathematics and Statistics, Xianning University, Xianning,
Hubei 437100, People's Republic of China
e-mail: junren751113@126.com

Y. Li
School of Chemistry and life science, Xianning University, Xianning,
Hubei 437100, People's Republic of China

on mathematical methods. Since the data cannot be measured by direct ways in many cases, we are always encountering inverse problems of deciding unknown sources and aquifer parameters. From the 1970s, there has been much research on such inverse and ill-posed problems in groundwater, and some wonderful results concerned with groundwater pollution source identification problems (see [5,6], for instance). The problem is ill-posed and any small change in the input data may result in a dramatic change in the solution. This inverse problem has been investigated in many papers. To the author's knowledge, the existence and uniqueness of the solution have been investigated in [2,10,11], the conditional stability and the data compatibility have been studied in [1,3,14], and numerical algorithms for the identification problem can be found in methods [7,8,12,13,16]. However, general regularization analysis for the problem is still limited. Recently, In [9], Trong et al. has studied regularization by the Fourier method, and given error estimates in the implicit representation. In this paper, we consider the following problem [9,14].

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x)\phi(t), & x \in \mathbb{R}, 0 < t < 1, \\ u(x,0) = 0, & x \in \mathbb{R}, \\ u(x,1) = g(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

where $\phi(t) = e^{-\lambda t}, \lambda > 0$, and satisfy $\phi(t) > C \geq C_0, C_0 > 0$. This system describes a heat process of radio isotope decay whose decay rate is λ.

We want to determine the source term $f(x)$ from the data $g(x)$. Since the data $g(x)$ is measured, there must exist measurement errors, and we actually have the measured data function $g^\delta \in L^2(\mathbb{R})$ which satisfies

$$\|g(\cdot) - g^\delta(\cdot)\| \leq \delta, \tag{1.2}$$

where $\|\cdot\|$ denotes the L^2-norm, and the constant $\delta > 0$ represents the noise level.

In general, for an ill-posed problem, the convergence rates of the regularization solution can only be given under a priori assumptions on the exact data. we will formulate such an a priori assumption in terms of the exact solution $f(x)$ by considering

$$\|f\|_p \leq M, p > 0. \tag{1.3}$$

where $\|\cdot\|_p$ denotes the norm in Sobolev space $H^p(R)$ defined by

$$\|f\|_p := \left(\int_{-\infty}^{+\infty} (1+\xi^2)^p |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}. \tag{1.4}$$

Let

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \tag{1.5}$$

be the Fourier transform of the function $f(x) \in L^2(\mathbb{R})$. Applying the Fourier transform with respect to variable x, problem (1.1) can be reformulated in the frequency space as follows:

$$\begin{cases} \hat{u}_t(\xi,t) + \xi^2 \hat{u}(\xi,t) = e^{-\lambda t}\hat{f}(\xi), & x \in \mathbb{R}, 0 < t < 1, \\ \hat{u}(\xi,0) = 0, & x \in \mathbb{R}, \\ \hat{u}(\xi,1) = \hat{g}(\xi), & x \in \mathbb{R}. \end{cases} \tag{1.6}$$

By elementary calculations we get

$$\hat{u}(\xi,t) = \frac{e^{-\xi^2 t}\left[e^{(\xi^2-\lambda)t} - 1\right]}{\xi^2 - \lambda}\hat{f}(\xi). \tag{1.7}$$

Substituting $\hat{u}(\xi,1) = \hat{g}(\xi)$ into (1.7), we have

$$\hat{g}(\xi) = \frac{e^{-\xi^2}\left[e^{(\xi^2-\lambda)} - 1\right]}{\xi^2 - \lambda}\hat{f}(\xi). \tag{1.8}$$

From (1.8), we can formulate the problem as an operator equation

$$\hat{A}\hat{f}(\xi) = \hat{g}(\xi), \tag{1.9}$$

where $\hat{A} = \frac{e^{-\xi^2}\left[e^{(\xi^2-\lambda)}-1\right]}{\xi^2-\lambda} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a self-adjoint multiplication operator.

Since the factor $\frac{e^{-\xi^2}\left[e^{(\xi^2-\lambda)}-1\right]}{\xi^2-\lambda}$ in (1.8) increases as $\xi^2 \to +\infty$, it is easy to see that the ill-posedness of Problem (1.1).

Although there exits a large of literature on inverse source problem for heat equation, most of them are devoted to numerical aspects, e.g. [16,18–20,22,24]. Few of them studied three important issues including existence, uniqueness and stability for this ill-posed problem.

In this study, we establish a stability estimate and give a generalized Tikhonov regularization method.

The paper is organized as follows. In Sect. 2, a conditional stability estimate for Problem (1.1) is proved; in Sect. 3, the error estimate for generalized Tikhonov regularization method is given; in Sect. 4, some numerical results are reported in order to show that the generalized Tikhonov regularization is suitable for solving this problem.

2 Conditional stability estimate

A conditional stability estimate for ill-posed problems tells how much any two solutions differ from each other when some error exists. Since Problem (1.1) is linear, we only need to derive the stability estimate on the solution near zero between the zero one.

Theorem 2.1 *Suppose that $f(x)$ is the solution of Problem (1.1), and (2.1) is satisfied, then the following estimate holds:*

$$\|f(\cdot)\|^2 \le e^2\|g(\cdot)\|^2 + \left(\frac{1}{e^{-\lambda} - e^{-1}}\right)^{\frac{2p}{p+2}} (\|f(\cdot)\|_p)^{\frac{4}{p+2}} \|g(\cdot)\|^{\frac{2p}{p+2}}. \qquad (2.1)$$

Proof Since $\hat{f}(\xi) = \frac{\xi^2-\lambda}{e^{-\xi^2}(e^{\xi^2-\lambda}-1)}\hat{g}(\xi)$, by Parseval identity we have

$$\|f(\cdot)\|^2 = \|\hat{f}(\cdot)\|^2 = \int\limits_{|\xi|\le 1} \left(\frac{\xi^2-\lambda}{e^{-\xi^2}(e^{\xi^2-\lambda}-1)}\hat{g}(\xi)\right)^2 |\hat{g}(\xi)|^2 d\xi$$
$$+ \int\limits_{|\xi|\ge 1} |\hat{f}(\xi)|^2 d\xi := A_1 + A_2.$$

Obviously,

$$\frac{\xi^2-\lambda}{e^{-\xi^2}(e^{\xi^2-\lambda}-1)} = \frac{(\xi^2-\lambda)e^{\xi^2}}{e^{\xi^2-\lambda}-1} \le e^{\xi^2}.$$

Hence,

$$A_1 \le e^2\|g(\cdot)\|^2.$$

Further by Hölder inequality,

$$\begin{aligned}
A_2 &= \int\limits_{|\xi|\ge 1} |\hat{f}(\xi)|^2 d\xi \\
&= \int\limits_{|\xi|\ge 1} \left[(1+\xi^2)^p|\hat{f}(\xi)|^2\right]^{\frac{2}{p+2}} \left[(1+\xi^2)^{-2}|\hat{f}(\xi)|^2\right]^{\frac{p}{p+2}} d\xi \\
&\le \left(\int\limits_{|\xi|\ge 1} (1+\xi^2)^p|\hat{f}(\xi)|^2 d\xi\right)^{\frac{2}{p+2}} \cdot \left(\int\limits_{|\xi|\ge 1} (1+\xi^2)^{-2}|\hat{f}(\xi)|^2 d\xi\right)^{\frac{p}{p+2}} \\
&\le \|f(\cdot)\|_p^{\frac{4}{p+2}} \cdot \left(\int\limits_{|\xi|\ge 1} (1+\xi^2)^{-2}\left|\frac{\xi^2-\lambda}{e^{-\xi^2}(e^{\xi^2-\lambda}-1)}\hat{g}(\xi)\right|^2 d\xi\right)^{\frac{p}{p+2}} \\
&\le \|f(\cdot)\Big|_p^{\frac{4}{p+2}} \cdot \left(\sup_{|\xi|\ge 1} \frac{(\xi^2-\lambda)^2}{(1+\xi^2)^2}\frac{1}{(e^{-\lambda}-e^{-\xi^2})^2}\right)^{\frac{p}{p+2}} \left(\int_{-\infty}^{+\infty} \hat{g}(\xi)\Big|^2 d\xi\right)^{\frac{p}{p+2}}
\end{aligned}$$

$$\le \|f(\cdot)\|_p^{\frac{4}{p+2}} \cdot \left(\frac{1}{e^{-\lambda}-e^{-1}}\|g(\cdot)\|\right)^{\frac{2p}{p+2}} = \left(\frac{1}{e^{-\lambda}-e^{-1}}\right)^{\frac{2p}{p+2}} \|f(\cdot)\|_p^{\frac{4}{p+2}} (\|g(\cdot)\|)^{\frac{2p}{p+2}}.$$

Hence we have (2.2). □

Remark 2.1 For given two functions $g_1(\cdot)$ and $g_2(\cdot)$, let $f_1(\cdot)$ and $f_2(\cdot)$ be the corresponding solutions, respectively, then

$$\|f_1(\cdot)-f_2(\cdot)\|^2 \le e^2\|g_1(\cdot)-g_2(\cdot)\|^2 + \left(\frac{1}{e^{-\lambda}-e^{-1}}\right)^{\frac{2p}{p+2}} (\|f_1(\cdot)-f_2(\cdot)\|_p)^{\frac{4}{p+2}} \|g_1(\cdot)-g_2(\cdot)\|^{\frac{2p}{p+2}}. \tag{2.2}$$

3 Generalized Tikhonov regularization and error estimate

Now we focus on the regularization problem. Here we use Tikhonov regularization method [21]. The method consists of looking for the solution for Problem (1.1), and minimizes the quadratic functional:

$$\|g(\cdot)-g_\delta\|^2 + \alpha\|f(\cdot)\|_p^2, \tag{3.1}$$

where $\alpha = \delta^2/M^2$. By Parseval's identity and (1.8) applied to $g(\cdot)$, the variational problem becomes

$$\|\frac{e^{-\xi^2}\left[e^{(\xi^2-\lambda)}-1\right]}{\xi^2-\lambda}\hat{f}(\xi)-g_\delta(\cdot)\|^2 + \alpha\|(1+\xi^2)^{p/2}\hat{f}(\cdot)\|^2. \tag{3.2}$$

Let $\hat{f}_\delta^\alpha(\cdot)$ be the solution of above problem, then it satisfies the Euler equation

$$\left(\left(\frac{e^{-\xi^2}\left[e^{(\xi^2-\lambda)}-1\right]}{\xi^2-\lambda}\right)^2 + \alpha(1+\xi^2)^p\right)\hat{f}_\delta^\alpha(\xi) = \frac{e^{-\xi^2}\left[e^{(\xi^2-\lambda)}-1\right]}{\xi^2-\lambda}\hat{g}_\delta(\xi). \tag{3.3}$$

The generalized Tikhonov regularization solution $\widehat{f_\delta^\alpha}(\cdot)$ in frequency domain can be given

$$\hat{f}_\delta^\alpha(\xi) = \frac{\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}}{1+\alpha(1+\xi^2)^p\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)^2}\hat{g}_\delta(\xi). \tag{3.4}$$

Now we will use the stability estimate (2.2) to derive the error estimate between the regularization solution and the exact one.

Theorem 3.1 *Suppose that $f(x)$ is the exact solution for Problem (1.1) with exact data $g(x)$, (3.4) is the generalized Tikhonov regularization solution with noisy data $g_\delta(x)$, (1.2) and (1.3) hold, then we have*

$$\|f_\delta^\alpha(\cdot) - f(\cdot)\|^2 \le e^2\delta^2 + \left(\frac{1}{e^{-\lambda} - e^{-1}}\right)^2 M^{\frac{4}{p+2}}\delta^{\frac{2p}{p+2}}. \tag{3.5}$$

Proof Via Parseval's identity, we have

$$\begin{aligned}
&\|\hat{f}_\delta^\alpha(\cdot) - \hat{f}(\cdot)\|_p \\
&= \left(\int_{-\infty}^{\infty}(1+\xi^2)^p\left|\frac{\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}}{1+\alpha(1+\xi^2)^p\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)^2}\hat{g}_\delta - \frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\hat{g}\right|^2 d\xi\right)^{1/2} \\
&\le \left(\int_{-\infty}^{\infty}(1+\xi^2)^p\left|\frac{\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}}{1+\alpha(1+\xi^2)^p\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)^2}\hat{g}_\delta\right.\right. \\
&\quad \left.\left.- \frac{\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}}{1+\alpha(1+\xi^2)^p\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)^2}\hat{g}\right|^2 d\xi\right)^{1/2} \\
&\quad + \left(\int_{-\infty}^{\infty}(1+\xi^2)^p\left|\frac{\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}}{1+\alpha(1+\xi^2)^p\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)^2}\hat{g} - \frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\hat{g}\right|^2 d\xi\right)^{1/2} \\
&\le \sup_{\xi\in R}\left|\frac{(1+\xi^2)^{p/2}\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}}{1+\alpha(1+\xi^2)^p\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)^2}\right| \|\hat{g}_\delta - \hat{g}\| \\
&\quad + \sup_{\xi\in R}\left|\frac{\alpha(1+\xi^2)^p\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)^2}{1+\alpha(1+\xi^2)^p\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)^2}\right| \|\hat{f}(\cdot)\|_p.
\end{aligned}$$

Since $l(\eta) := \eta(1+\eta^2)^{-1}, \eta > 0$ has a maximum $1/2$,

$$\sup_{\xi\in R}\left|\frac{(1+\xi^2)^{p/2}\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)}{1+\alpha(1+\xi^2)^p\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)^2}\right| \le \frac{1}{2}(\sqrt{\alpha})^{-1} = \frac{1}{2}\frac{M}{\delta},$$

$$\sup_{\xi\in R}\left|\frac{\alpha(1+\xi^2)^p\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)^2}{1+\alpha(1+\xi^2)^p\left(\frac{\xi^2-\lambda}{e^{-\lambda}-e^{-\xi^2}}\right)^2}\right| < 1.$$

Thus

$$\|\hat{f}_\delta^\alpha(\cdot) - \hat{f}(\cdot)\|_p \le \frac{3}{2}M. \tag{3.6}$$

Hence according to (2.2) we have the error estimate

$$\begin{aligned}\|f_\delta^\alpha(\cdot) - f(\cdot)\|^2 &\le e^2\delta^2 + \left(\frac{1}{e^{-\lambda} - e^{-1}}\right)^{\frac{2p}{p+2}} (1.5M)^{\frac{4}{p+2}} \delta^{\frac{2p}{p+2}} \\ &\le e^2\delta^2 + \left(\frac{1}{e^{-\lambda} - e^{-1}}\right)^2 M^{\frac{4}{p+2}} \delta^{\frac{2p}{p+2}}. \end{aligned} \tag{3.7}$$

□

Remark 3.1 Now we seek a solution in frequency domain for Problem (1.1) in $L^2[-\xi_{\max}, \xi_{\max}]$ where $\xi_{\max}$ is a positive number which plays a role of regularization parameter. This is the so-called Fourier regularization method [15]. Similar to generalized Tikhonov regularization, we can easily obtain the error estimate for Fourier regularization method.

4 Numerical tests

In this section we give two examples to verify the effect of the proposed methods.

Example 1 The pair of functions [15] (in [15], $(t+1)^{2/3}$ should be $(t+1)^{3/2}$)

$$\begin{aligned} u(x,t) &= e^{-\lambda} \frac{x}{(t+1)^{3/2}} \exp\left(-\frac{x^2}{4(t+1)}\right) - x\exp\left(-\frac{x^2}{4}\right), \\ f(x) &= e^{-\lambda}\left(\frac{x^3}{4} - \frac{3x}{2}\right)\exp\left(-\frac{x^2}{4}\right) \end{aligned} \tag{4.1}$$

is the exact solution of Problem (1.1) with data

$$g(x) = e^{-\lambda}\frac{x}{2^{3/2}}\exp\left(-\frac{x^2}{8}\right) - x\exp\left(-\frac{x^2}{4}\right). \tag{4.2}$$

Similar to [15], we will do the numerical tests in the interval $x \in [-10, 10]$, the noise level δ is computed in the same way. The computed errors are defined by

$$l^2\ norm\ error: \quad E(f) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(f(x_i) - f_r(x_i))^2}. \tag{4.3}$$

$$relative\,error: \quad ER(f) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(f(x_i) - f_r(x_i))^2} \Bigg/ \sqrt{\frac{1}{n}\sum_{i=1}^{n}(f(x_i))^2}, \tag{4.4}$$

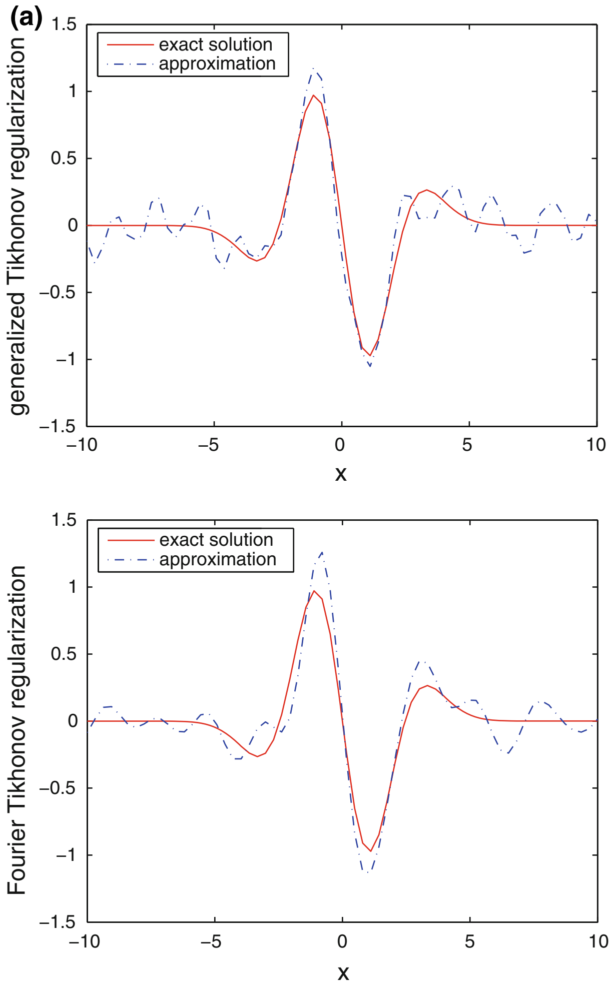

Fig. 1 $p = 1/2, \delta = 0.1$, (**a**): generalized Tikhonov regularization with $RE(f_{GTK}) = 39\%$; (**b**): Fourier regularization with $RE(f_{FOU}) = 39\%$

where $x_i, i = 1, 2, \ldots, n$ are the test points and in computation, we also take $n = 64$, $f_r(\cdot)$ denotes the regularization solution. According to [15], $\|f\|_{1/2} = 1.96$ and $\|f\|_2 = 4.45$. In numerical tests, the regularization parameter α for generalized Tikhonov regularization is selected by $\alpha = \frac{\delta^2}{M^2}$, the regularization parameter $\xi_{\max}$ for Fourier regularization is selected by $\xi_{\max} = \left(\frac{M}{\delta}\right)^{\frac{1}{p+2}}$ according to [15].

Throughout this section, f_{GTK} and f_{FOU} represent the generalized Tikhonov regularization method and Fourier regularization method [15], respectively.

From Fig. 1, we can conclude that the the approximation effect of generalized Tikhonov regularization and the Fourier Tikhonov regularization is comparable. From

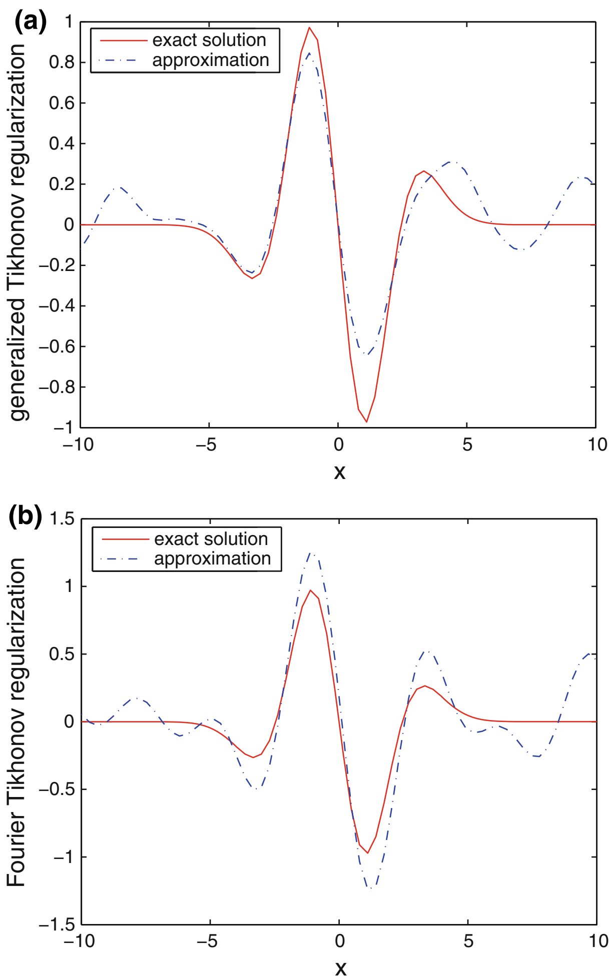

Fig. 2 $p = 2, \delta = 0.3$, (**a**): generalized Tikhonov regularization with $RE(f_{GTK}) = 44\%$; (**b**): Fourier regularization with $RE(f_{FOU}) = 57\%$

Fig. 2, we can see that for larger error level δ, the approximation effect of generalized Tikhonov regularization is better.

Example 2 The pair of functions

$$\begin{aligned} u(x,t) &= e^{-\lambda}(1 - e^{-\pi^2 t})\sin(\pi x), \\ f(x) &= e^{-\lambda}\pi^2 \sin(\pi x) \end{aligned} \tag{4.5}$$

is the exact solution of Problem (1.1) with data

$$g(x) = e^{-\lambda}(1 - e^{-\pi^2})\sin(\pi x). \tag{4.6}$$

Table 1 $ER(f)$ with p

p	0.5	1	1.5	2	2.5
$ER(f_{GTK})$	0.219	0.138	0.135	0.304	0.583

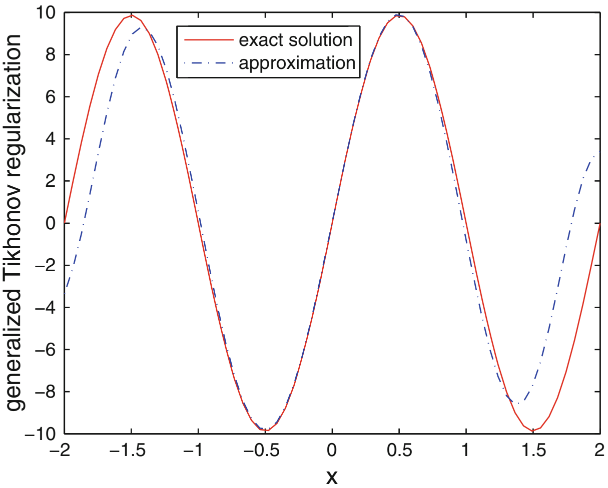

Fig. 3 $p = 1, \delta = 0.006, ER(f_{GTK}) = 14\%$

For this example, the value of M is not easy for one to obtain. In practice, it is hard for one to the value of M without having exact solution. Thus we try to take $M = 1$ for choosing regularization parameter α,i.e., $\alpha = \delta^2$. In this example, let p vary, we observe how the M influences the numerical results. See Table 1 with $\delta = 0.006, n = 64$, the interval $x \in [-10, 10]$.

From Table 1, we can conclude that p has some regularization effect. How to select p is a challenge because we don't know the relation between p and M explicitly. Figure 3 shows the approximation effect for generalized Tikhonov regularization method with test interval $x \in [-2, 2]$.

5 Concluding remark

The issue of conditional stability on inverse ill-posed problems is very important. In this paper, we give an 'optimal' stability estimate. Based on the obtained estimate, the error estimate between Tikhonov regularization solution and the exact solution can be derived. The method in this note can be applied to the other ill-posed problems [17,23].

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